Sturm-Liouville Theory (part 2)

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda w(x)y = 0 \qquad a \le x \le b$$

Subject to $\quad \alpha_1 y(a) + \alpha_2 y'(a) = 0$

$\qquad\qquad\quad \beta_1 y(b) + \beta_2 y'(b) = 0$

the SL problem is said to be <u>regular</u> if on $a \le x \le b$,

$p(x)$, $p'(x)$, $q(x)$, $w(x)$ are <u>continuous</u> and

$p(x) > 0$, $w(x) > 0$

→ infinitely many <u>eigenvalues</u> λ that form an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \qquad \lambda_n \to \infty \text{ as } n \to \infty$$

(these could be negative but if $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all nonnegative, then λs are also nonnegative)

→ each λ_n is paired with an eigenfunction y_n and these eigenfunctions are <u>mutually orthogonal</u> with respect to the weight function $w(x)$

$$\int_a^b y_n y_m w(x)\,dx = 0 \quad \text{if } n \ne m$$

example $y'' + \lambda y = 0 \qquad 0 < x < L$ looks just like

$y(0) = 0 \qquad\qquad X'' + \lambda X = 0$

$y(L) = 0 \qquad\qquad X(0) = 0$

$X(L) = 0$

$p = 1,\ q = 0,\ w = 1$

$\alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 1,\ \beta_2 = 0$

so, this is a regular SL problem w/ nonnegative λs

we know $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ nonnegative (agrees w/ SL theory)

$$y_n = \sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L y_n y_m w(x)\,dx = \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0$$

(as predicted by SL theory)

the nonnegative λ is the reason why we assumed the separation constant to be $-\lambda$

$$\frac{X''}{X} = \frac{T'}{KT} = -\lambda$$

→ Fourier solution is just a special case of SL problem

<u>example</u> $y'' + \lambda y = 0 \qquad 0 < x < L \qquad p=1,\ q=0,\ w=1$

$y(0) = 0 \qquad \alpha_1 = 1,\ \alpha_2 = 0$

$hy(L) - y'(L) = 0 \qquad h > 0 \qquad \beta_1 = h > 0,\ \beta_2 = -1$

Since not all α, β are nonnegative, the eigenvalues λs are not always nonnegative

$\longrightarrow$ negative eigenvalues are possible

let's investigate the (possibly) negative eigenvalues

$y'' + \lambda y = 0$

$\lambda < 0$, for convenience let $\lambda = -k^2$ $(k > 0)$

(k^2 avoids $\sqrt{\lambda}$ in solution
convenience only)

$y'' - k^2 y = 0$

$y = c_1 e^{kx} + c_2 e^{-kx}$

or $y = A\cosh(kx) + B\sinh(kx)$

} choose whichever is more convenient w/ give BCs w/ $y(0)=0$ and $y=c_1e^{kx}+c_2e^{-kx}$

we see $0=c_1+c_2$ ok, but it would be nice if one of them is 0

w/ $y(0)=0$ and $y=A\cosh(kx)+B\sinh(kx)$

we see $0=A \rightarrow y=B\sinh(kx)$

$\lambda=-k^2$

(eigenfunctions are $y_n=\sinh(kx)$)

use $hy(L)-y'(L)=0$ to find k (and therefore λ)

$y=B\sinh(kx)$

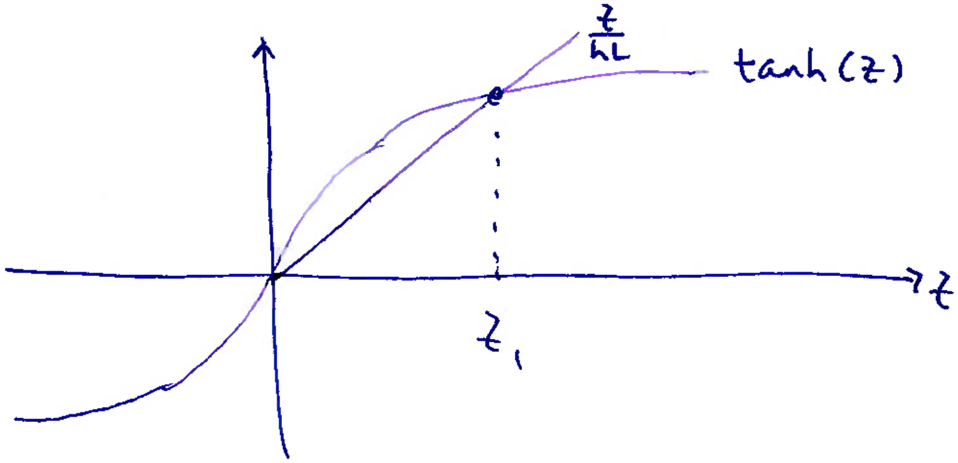
$y'=Bk\cosh(kx)$

$hB\sinh(kL)-Bk\cosh(kL)=0$ $B\neq0,\ L\neq0,\ h\neq0$

$\tanh(kL)=\frac{k}{h}$ solve for k

$\tanh(kL)=\frac{kL}{hL} \rightarrow$ solve $\tanh(z)=\frac{z}{hL}$ $z=kL$

intersection of $\tanh(z)$ and line $\frac{z}{hL}$



at most one intersection (if hL is small the line is too steep to intersect $\tanh(z)$)

at most one intersection → at most one negative eigenvalue

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

negative — must be ≥ 0

this negative λ is the <u>smallest</u> eigenvalue (no largest λ)

$$\lambda_1 = -k^2 = -\left(\frac{z_1}{L}\right)^2$$

corresponding eigenfunction: $y_1 = \sinh\left(\frac{z_1}{L}x\right)$

next: is $\lambda = 0$ and eigenvalue

then find $\lambda > 0$

let's revisit the heat exchange problem from last time

$$u_t = k u_{xx} \qquad 0 < x < L$$

$$u(0,t) = 0$$

$$u_x(L,t) = -h u(L,t) \qquad h > 0$$

$$u(x,0) = 100$$ initially heated to 100 uniformly

last time we solved the spatial problem: $y'' + \lambda y = 0$

$$\lambda_n = \frac{z_n^2}{L^2}$$

z_n: positive intersections of $\tan(z)$ and $-\frac{z}{hL}$

solve the time problem as usual

then the general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n t} \sin(\sqrt{\lambda_n}\, x)$$

$u(x,0) = 100$

$$100 = \sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}\, x)$$

NOT a sine series since

λ is not integer multiple of $\frac{\pi}{L}$

So, $C_n \neq \frac{2}{L}\int_0^L 100 \sin(\frac{n\pi}{L}x)\,dx$

we can still use the orthogonality of eigenfunctions (guaranteed by SL theory) to find C_n

$$\int_0^L y_n y_m w(x)\,dx = 0 \qquad n \neq m$$

here, $w(x) = 1$

$y_n = \sin(\sqrt{\lambda_n}\, x)$

find C_n:

$$100 \sin(\sqrt{\lambda_m}\, x) = \sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}\, x) \sin(\sqrt{\lambda_m}\, x)$$

integrate over $0 < x < L$

$$\int_0^L 100 \sin(\sqrt{\lambda_m}\, x)\,dx = \sum_{n=1}^{\infty} \underbrace{\int_0^L c_n \sin(\sqrt{\lambda_n}\, x) \sin(\sqrt{\lambda_m}\, x)\,dx}_{0 \text{ except } n=m}$$

$$\int_0^L 100 \sin(\sqrt{\lambda_n}\, x)\,dx = \int_0^L c_n \sin^2(\sqrt{\lambda_n}\, x)\,dx$$

$$\boxed{c_n = \frac{\int_0^L 100 \sin(\sqrt{\lambda_n}\, x)\,dx}{\int_0^L \sin^2(\sqrt{\lambda_n}\, x)\,dx}}$$

if $\lambda_n = \frac{n^2\pi^2}{L^2}$ it reduces to $\frac{2}{L}\int_0^L 100 \sin(\frac{n\pi}{L} x)\,dx$